A simple-minded Lagrangian approach seems to produce normal diffusion. Also, if $l_o/\delta x = \kappa/c_s \lesssim 1$, and at the same time $T \ll \alpha(\delta x)^2$ (small amplitude fluctuatios), we would find $T/(\rho c_s \kappa) \ll 1$, meaning that the diffusivity correction is small. Back to Eq. (267) this implise $S_2(t) \ll M_2(t)$: the direct deformation field ballistic contribution to heat transport is small. Smallness of the ratio S_2/M_2 has the same origin of the smallness of the bare coupling constant g_0, that is the small deformation regime considered. The fact that the bare coupling is small, however, did not prevent the renormalization of κ to diverge at sufficiently large scales. We cannot disregard the contribution from the correlation of small scale fluctuations of f and $\dot{q}$ to the large scale transport of $\bar{f}$.

8 Irreversible stochastic dynamics

We say that a dynamics is irreversible if it does not satisfy detailed balance. This requires a Langevin dynamics with more than one degrees of freedom. Such a Langevin dynamics can be obtained from a Markovian jump dynamics $x_n \to x_{n\pm1}$, with the coefficients in the Langevin equation $\mathrm{d}x = a(x)\mathrm{d}t + b(x)\mathrm{d}B$ expressed in terms of transition probabilities as

$$\begin{aligned} a(x)\delta t &= [P(x_n \to x_{n+1}) - P(x_{n+1} \to x_n)]\delta x, \\ (b(x))^2\delta t &= [P(x_n \to x_{n+1}) + P(x_{n+1} \to x_n)]\delta x^2, \end{aligned} \tag{269}$$

where $x = (x_n + x_{n+1})/2$. Since the jumps occur between adjacent links in the chain, detailed balance is trivially satisfied, which implies that the continuum dynamics is reversible. The vice versa is not true; an irreversible microscopic dynamics may lead to a macroscopic reversible dynamics. To see this, just add an intermediate step y_n in the jump process, with $x_n < y_n < x_{n+1}$, and still $x = (x_n + x_{n+1})/2$, in such a way that y_n does not enter the dynamics in the continuum, and impose the rule on the transition probabilities

$$\begin{aligned} \bar{P}(x_n)P(x_n \to y_n) &= \bar{P}(y_n)P(y_n \to x_{n+1}) = \bar{P}(x_{n+1})\bar{P}(x_{n+1} \to x_n); \\ P(x_n \to x_{n+1}) &= P(x_{n+1} \to y_n) = P(y_n \to x_n) = 0, \end{aligned} \tag{270}$$

which does not satisfy detailed balance. The coarse graining in Eq. (269) is replaced by

$$\begin{aligned} a(x)\delta t &= [P(x_n \to y_n) - P(x_{n+1} \to x_n)]\delta x, \\ (b(x))^2\delta t &= [P(x_n \to y_n) + P(x_{n+1} \to x_n)]\delta x^2, \end{aligned} \tag{271}$$

which however leads to the same Langevin dynamics $\mathrm{d}x = a(x)\mathrm{d}t + b(x)\mathrm{d}B$.

We recall the conditions for macroscopic reversibility for a generic multivariate Langevin dynamics $\mathrm{d}x_i = a_i(\mathbf{x})\mathrm{d}t + b_{ij}(\mathbf{x})\mathrm{d}B_j$ and define $D_{ij}(\mathbf{x}) = b_{ik}(\mathbf{x})b_{kj}(\mathbf{x})$. We must have

$$\begin{aligned} 0 &= \partial_t[P(\mathbf{x},t|\mathbf{y},0)\bar{P}(\mathbf{y}) - P(\mathbf{y},t|\mathbf{x},0)\bar{P}(\mathbf{x})] \\ &= [\hat{L}^+(\mathbf{x})P(\mathbf{x},t|\mathbf{y},0)\bar{P}(\mathbf{y}) - \hat{L}^+(\mathbf{y})P(\mathbf{y},t|\mathbf{x},0)\bar{P}(\mathbf{x})], \end{aligned} \tag{272}$$

where $\hat{L}^+(\mathbf{x}) = -\partial_{x_i}(a_i(\mathbf{x})\ .\ )+\partial_{x_i}\partial_{x_j}(D_{ij}(\mathbf{x})\ .\ )$ is the Fokker-Planck operator. Integrating over generic functions $f(\mathbf{x})$ and $g(\mathbf{y})$ gives

$$\begin{aligned} 0 &= \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ f(\mathbf{y})g(\mathbf{x})[\hat{L}^+(\mathbf{x})P(\mathbf{x},t|\mathbf{y},0)\bar{P}(\mathbf{y}) - \hat{L}^+(\mathbf{y})P(\mathbf{y},t|\mathbf{x},0)\bar{P}(\mathbf{x})] \\ &= \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ f(\mathbf{y})[P(\mathbf{x},t|\mathbf{y},0)\bar{P}(\mathbf{y})\hat{L}(\mathbf{x})g(\mathbf{x}) - \hat{L}^+(\mathbf{y})P(\mathbf{y},t|\mathbf{x},0)\bar{P}(\mathbf{x})g(\mathbf{x})], \end{aligned} \tag{273}$$

where $\hat{L}(\mathbf{x}) = a_i(\mathbf{x})\partial_{x_i} + D_{ij}(\mathbf{x})\partial_{x_i}\partial_{x_j}$ is the adjoint of the Fokker-Planck operator. Taking the $\mathbf{x} \to \mathbf{y}$ limit we get

$$\begin{aligned} 0 &= \int \mathrm{d}\mathbf{x}\mathrm{d}\mathbf{y}\ f(\mathbf{y})[\delta(\mathbf{x}-\mathbf{y})\bar{P}(\mathbf{y})\hat{L}(\mathbf{x})g(\mathbf{x}) - \hat{L}^+(\mathbf{y})\delta(\mathbf{x}-\mathbf{y})\bar{P}(\mathbf{x})g(\mathbf{x})] \\ &= \int \mathrm{d}\mathbf{x}\ f(\mathbf{x})\bar{P}(\mathbf{x})\hat{L}g(\mathbf{x}) - \int \mathrm{d}\mathbf{y}\ f(\mathbf{y})\hat{L}^+(\mathbf{y})\int \mathrm{d}\mathbf{x} g(\mathbf{x})\bar{P}(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}) \\ &= \int \mathrm{d}\mathbf{x}\ f(\mathbf{x})\Big[\bar{P}(\mathbf{x})\hat{L}g(\mathbf{x}) - \hat{L}^+ g(\mathbf{x})\bar{P}(\mathbf{x})\Big]. \end{aligned} \tag{274}$$

The relation can be rewritten in the form

$$\int \mathrm{d}\mathbf{x}\ \bar{P}(\mathbf{x})\Big[f(\mathbf{x})\hat{L}g(\mathbf{x}) - g(\mathbf{x})\hat{L}f(\mathbf{x})\Big] = 0, \tag{275}$$

which tells us that $\hat{L}$ must be self-adjoint with respect to the L^2 norm with weight $\bar{P}(\mathbf{x})$. Equation (275) implies

$$\int \mathrm{d}\mathbf{x}\ \bar{P}(\mathbf{x})\Big[f(\mathbf{x})\langle\dot{g}|\mathbf{x}\rangle - g(\mathbf{x})\langle\dot{f}|\mathbf{x}\rangle] = 0 \Rightarrow \langle f\dot{g}\rangle = \langle g\dot{f}\rangle, \tag{276}$$

which has as special case,

$$\langle x_i\dot{x}_j\rangle = \langle x_j\dot{x}_i\rangle. \tag{277}$$

The above can clearly be seen as a precursor of Onsager's relations.

We can obtain reversibility conditions on the drift and the diffusion in the Fokker-Planck operator directly from Eq. (274), by writing explicitly

$$\begin{aligned} 0 &= \int \mathrm{d}\mathbf{x}\ f[\bar{P}(a_i\partial_i + D_{ij}\partial_i\partial_j)g + \partial_i(a_i\bar{P}g) - \partial_i\partial_j(D_{ij}\bar{P}g)] \\ &= \int \mathrm{d}\mathbf{x}\ (f\partial_i g)[a_i\bar{P} - \partial_j(D_{ij}\bar{P})], \end{aligned} \tag{278}$$

where use has been made of $\hat{L}^+\bar{P} = 0$. We obtain finally the condition that the probability current at stationarity must be zero

$$a_i\bar{P} - \partial_j(D_{ij}\bar{P}) = 0, \tag{279}$$

which is tantamount to a condition of detailed balance in the continuum. We can rearrange the equation to give

$$D^{-1}_{jk}(a_k - \partial_l D_{lk}) = \partial_j \ln \bar{P}, \tag{280}$$

which, in order to be satisfied, requires the LHS to be itself a gradient. We obtain finally

$$a_k = \partial_l D_{lk} + D_{lk}\partial_l \ln \bar{P}. \tag{281}$$

We can calculate the degree of asymmetry in the time correlation $\langle g\dot{f}\rangle$ from the stationary current. Comparing Eqs. (276) and (278) we find

$$\langle f\dot{g}\rangle - \langle \dot{f}g\rangle = \langle f J_l \partial_l g\rangle \tag{282}$$

which has as special case

$$\langle x_i \dot{x}_j\rangle - \langle \dot{x}_i x_j\rangle = \langle x_i J_j\rangle. \tag{283}$$

The reversible or irreversible nature of the dynamics has consequences on the way the coefficients in the Fokker-Planck equation and in the associated SDE transform under time reversal. In the case of a reversible dynamics, we know that that the drift is symmetric under time reversal

$$a_{forward} = \frac{\langle [x(t+\Delta t) - x] | x, t\rangle}{\Delta t} = \frac{\langle [x(t-\Delta t) - x] | x, t\rangle}{\Delta t} = a_{backward}. \tag{284}$$

This corresponds to forward and backward SDE's of identical form $\mathrm{d}x = a\mathrm{d}t + \mathrm{d}\xi$ for $\mathrm{d}t > 0$ and $\mathrm{d}t < 0$, and identical forward and backward Fokker-Planck equations $\partial_P = \partial_x[-aP + \partial_x(DP)]$ and $\partial_{\hat{t}} P = \partial_x[-aP + \partial_x(DP)]$, $\hat{t} = t_f - t$.

Reversibility breakup is associated with the presence of a stationary current. Time reversal of a stationary irreversible distribution produces an identical stationary distribution but with stationary current of opposite sign (a current flowing clockwise forward in time, flows counterclockwise seen backward in time). This requires the presence of a drift component that is antisymmetric under time reversal. The stationary current obtained from the forward Fokker-Planck equation is $J_i = (-a_i + \partial_j D_{ij} + D_{ij}\partial_j \ln \bar{P})\bar{P}$, where $\partial_j J_i = 0$, which can be rewritten, by defining $\hat{J}_i = J_i/\bar{P}$,

$$-a_i + \partial_j D_{ij} + D_{ij}\partial_j \ln \bar{P} = \hat{J}_i. \tag{285}$$

Let us indicate with $2a^-_i$ the drift contribution that must be subtracted to a_i to reverse sign to J_i for fixed $\bar{P}$:

$$-a_i + 2a^-_i + \partial_j D_{ij} + D_{ij}\partial_j \ln \bar{P} = -\hat{J}_i \tag{286}$$

Taking half the sum of Eqs. (285) and (286) gives

$$a_i - a^-_i - \partial_j D_{ij} = D_{ij}\partial_j \ln \bar{P}, \tag{287}$$

which allows us to identify with

$$\bar{a}_i = a_i - a_i^- = \partial_j D_{ij} + D_{ij}\partial_j \ln \bar{P} \quad \text{and} \quad a_i^- = a_i - \bar{a}_i, \tag{288}$$

respectively, the reversible and the irreversible component of the drift.

Part II
Systems with mixed boundary conditions in time

9 On the establishment of a causality principle

We say that a system has a causal dynamics if its response to a perturbation by an external agent does not anticipate the perturbation (past and future defined relative to the arrow of time of the universe). The fact that the agent can be considered external is crucial: it basically means that the past history of the system does not influence perturbation enactment.

The following example illustrates the situation.

Suppose our system is described by the Langevin equation $\dot{x} + \Gamma x = \xi + f$, where ξ is the white noise and $f(t)$ is the perturbation. By choosing $f(t) = X\delta(t-T)$ with X large, fixed independently of $x(t)$, with the system originally in equilibrium conditions, we find $x(t) \simeq X\exp(-\Gamma(t-T))$ for $t > T$.[4] The perturbation follows the forcing, as expected in a causal dynamics, and dissipates in the future (it decays for $t \to \infty$). The system and the agent are uncorrelated in the past and become correlated in the future.

Now suppose we allow the perturbation to depend on the past history of the system, we could arrange things in such a way, say, that the perturbation is switched on only if $x(t) = X + \tilde{x}$, where $\tilde{x}$ is a random variable with $\sigma_{\tilde{x}} = \sigma_x$ and $f = -X\delta(t-T)$. The system would evolve along an atypical trajectory for $t < T$, $x(t) = X\exp[\Gamma(t-T)]$, and be in equilibrium for $t > T$—a state of affair undistinguishable from that of an anticausal-antidissipative response, with system and agent uncorrelated in the future and correlated in the past.

This suggests us that causality and dissipation are not intrinsic properties of the system, instead, they are consequence of the fact that we experimentally realize the perturbation without sampling. We surmise that causality and dissipation are not intrinsic characteristic of the system, rather, they are consequence of the boundary conditions imposed on the dynamics, consistent with the fact that the equilibrium statistics, away from initial or final conditions, is time-reversible.

[4]We choose to work in a large deviation regime only for simplicity, which allows us to focus on the leading trajectory, instead of having to deal with an ensemble of trajectories.