5 The Martin-Siggia-Rose formalism

We consider first the case of an additive SDE

$$\mathrm{d}x = a(x)\mathrm{d}t + D^{1/2}\mathrm{d}W. \tag{69}$$

We discretize time and for simplicity take $D = 1$. The evolution of the PDF $\rho(x,t)$ is given by

$$\begin{aligned}
\rho(x, t+\Delta t) &= \frac{1}{\sqrt{2\pi\Delta t}} \int \mathrm{d}x' \int \frac{\mathrm{d}k}{2\pi} \int \mathrm{d}\Delta W \;\; \exp\Big\{ -\mathrm{i}k[x - x' - a(x')\Delta t - \Delta W] \\
&- \frac{|\Delta W|^2}{2\Delta t}\Big\}\rho(x', t) \\
&= \int \mathrm{d}x' \int \frac{\mathrm{d}k}{2\pi} \;\; \exp\Big\{ -\mathrm{i}k[x - x' - a(x')\Delta t] - \frac{\Delta t}{2}k^2\Big\}\rho(x', t) \\
&= \frac{1}{\sqrt{2\pi\Delta t}} \int \mathrm{d}x' \;\; \exp\Big\{ - \frac{|x - x' - a(x')\Delta t|^2}{2\Delta t}\Big\}\rho(x', t).
\end{aligned} \tag{70}$$

With some care, we can derive the Fokker-Planck equation ($y = x - x' - a(x)\Delta t$)

$$\begin{aligned}
&\rho(x, t+\Delta t) \simeq \frac{1}{\sqrt{2\pi\Delta t}} \int \mathrm{d}x' \;\; \exp\Big\{ - \frac{|x - x' - [a(x) + (x'-x)a'(x)]\Delta t|^2}{2\Delta t}\Big\} \\
&\times\rho(x', t) \simeq \frac{1}{\sqrt{2\pi\Delta t}} \int \mathrm{d}y \;\; \exp(-y^2/(2\Delta t))\Big\{\rho(x,t)[1 - (y - a(x)\Delta t)a'(x)y] \\
&+(y - a(x)\Delta t)\rho'(x,t) + \frac{1}{2}(y - a(x)\Delta t)^2\rho''(x,t)\Big\} \\
&\simeq (1 - a'(x)\Delta t)\rho(x,t) - \Delta t a(x)\rho'(x,t) + \frac{1}{2}\Delta t\rho''(x,t),
\end{aligned} \tag{71}$$

and we immediately find a prescription problem: we cannot approximate in the equation $a(x') \simeq a(x)$. Taking the continuous limit,

$$\partial_t\rho(x,t) = -\partial_x(a(x)\rho(x,t)) + \frac{1}{2}\partial_x^2\rho(x,t) := \hat{H}\rho(x,t), \tag{72}$$

and we find the Green function

$$G(x, x'; \Delta t) \;\; \equiv \;\; \Big\langle x\Big|\mathrm{e}^{\hat{H}\Delta t}\Big|x'\Big\rangle = \frac{1}{\sqrt{2\pi\Delta t}}\exp\Big\{ - \frac{|x - x' - a(x')\Delta t|^2}{2\Delta t}\Big\}. \tag{73}$$

From Eq. (70) we can obtain three equivalent path-integral representations:

$$
\begin{aligned}
\rho(x,t) &= \mathcal{N}\int [\mathrm{d}x]\ \delta(x-x_{n+1})\exp\Big\{-\sum_{i=0}^{n}\frac{|x_{i+1}-x_i-a(x_i)\Delta t|^2}{2\Delta t}\Big\}\rho(x_0,t) \\
&= \mathcal{N}\int [\mathrm{d}x][\mathrm{d}k]\ \delta(x-x_{n+1}) \\
&\times\ \exp\Big\{\sum_{i=0}^{n}\Big[-\mathrm{i}k_{i+1}[x_{i+1}-x_i-a(x_i)\Delta t]-\frac{\Delta t}{2}k_{i+1}^2\Big]\Big\}\rho(x_0,t) \\
&= \mathcal{N}\int [\mathrm{d}x][\mathrm{d}k][\mathrm{d}W]\ \delta(x-x_{n+1})\exp\Big\{\sum_{i=0}^{n}\Big[-\mathrm{i}k_{i+1}[x_{i+1}-x_i \\
&-\ a(x_i)\Delta t-\Delta W_i]-\frac{|\Delta W_i|^2}{2\Delta t}\Big]\Big\}\rho(x_0,0),
\end{aligned}
\tag{74}
$$

where $[\mathrm{d}x]\equiv\prod_{i=0}^{n+1}\mathrm{d}x_i$, $[\mathrm{d}k]\equiv\prod_{i=1}^{n+1}\mathrm{d}k_i$ and $[\mathrm{d}W]\equiv\prod_{i=0}^{n}\mathrm{d}W_i$.

The variable $k_i\equiv k(t_i)$ parameterizes the response of the system to an external perturbation. We can perturb the dynamics by substituting $\Delta W_i\to\Delta W_i+\xi_i\Delta t$ in the expression $[x_{i+1}-x_i-a(x_i)\Delta t-\Delta W_i]$, which appears in the last line of Eq. (74), and the operation is clearly equivalent to replacing $x_{i+1}\to x_{i+1}-\xi_i\delta t$ in the same expression. The perturbed PDF is

$$
\rho_\xi(x,t)\simeq\rho(x,t)+\sum_{i}^{t_i<t}\frac{\partial\rho_\xi(x,t)}{\partial\xi_i}\Big|_{\xi=0}\xi_i\to\int_0^t\mathrm{d}\tau\frac{\delta\rho_\xi(x,t)}{\delta\xi(\tau)}\Big|_{\xi=0}\xi(\tau),
\tag{75}
$$

where

$$
\begin{aligned}
\frac{\partial\rho_\xi(x,t)}{\partial\xi_i}|_{\xi=0} &= -\frac{\partial\rho(x,t)}{\partial x_{i+1}}=\mathcal{N}\int[\mathrm{d}x][\mathrm{d}k]\ \delta(x-x_{n+1})\ \mathrm{i}k_{j+1} \\
\times\exp&\Big\{\sum_{i=0}^{n}\Big[-\mathrm{i}k_{i+1}[x_{i+1}-x_i-a(x_i)\Delta t]-\frac{\Delta t}{2}k_{i+1}^2\Big]\Big\}\bar\rho(x_0),
\end{aligned}
\tag{76}
$$

and for simplicity we have taken initial statistical equilibrium conditions $\rho(x_0,t_0)=\bar\rho(x_0)$. Higher orders in the Taylor expansion of the response to a finite perturbation can be obtained by including additional factors ik in the path-integral in Eq. (76). We thus find the response function

$$
\frac{\delta\langle x(t)\rangle_\xi}{\delta\xi(\tau)}\Big|_{\xi=0}=\mathrm{i}\langle k(\tau)x(t)\rangle.
\tag{77}
$$

We can use the variable k to express space derivatives of the probability:

$$
\begin{aligned}
\frac{\partial\rho(x,t)}{\partial x} &= -\mathcal{N}\int[\mathrm{d}x][\mathrm{d}k]\ \delta(x-x_{n+1})\mathrm{i}k_{n+1} \\
&\times\ \exp\Big\{\sum_{i=0}^{n}\Big[-\mathrm{i}k_{i+1}[x_{i+1}-x_i-a(x_i)\Delta t]-\frac{\Delta t}{2}k_{i+1}^2\Big]\Big\}\bar\rho(x_0),
\end{aligned}
$$

and we notice the opposite sign with respect to Eq. (76). We can then express the"Hamiltonian" in Eq. (72) in operatorial form

$$\hat{H} = H(-\partial_x, x) \equiv H(\mathrm{i}\hat{k}, \hat{x}) = \mathrm{i}\hat{k}a(\hat{x}) - \frac{\hat{k}^2}{2}. \tag{78}$$

The non-commutativity of operators $\hat{k}$ and $\hat{x}$ is just another manifestation of the prescription problems of stochastic calculus.

Working with the operator $\hat{k}$ allows us to recover the path-integral expression

$$\begin{aligned}\rho(x,t) &= \Big\langle x \Big| \mathrm{e}^{\hat{H}t} \Big| \rho_0 \Big\rangle \simeq \Big\langle x \Big| (1+\hat{H}\Delta t)^{t/\Delta t} \Big| \rho_0 \Big\rangle \\ &= \Big\langle x \Big| \ldots \Big| x_1 \Big\rangle \Big\langle x_1 \Big| k_1 \Big\rangle \Big\langle k_1 \Big| [1+\hat{H}\Delta t] \Big| x_0 \Big\rangle \Big\langle x_0 \Big| \rho_0 \Big\rangle. \end{aligned} \tag{79}$$

where, in the expressions $|x\rangle\langle x|$ and $|k\rangle\langle k|$, there is an implied integral over x and k respectively (recall that in our convention, $\langle x|k\rangle = \mathrm{e}^{-\mathrm{i}kx}$). By exploiting the relation

$$\begin{aligned}\Big\langle x_1 \Big| k_1 \Big\rangle \Big\langle k_1 \Big| [1+\hat{H}\Delta t] \Big| x_0 \Big\rangle &= \exp\{-\mathrm{i}k_1(x_1-x_0)\}\Big[1+\Delta t\Big(\mathrm{i}k_1 a(x_0) - \frac{k_1^2}{2}\Big)\Big] \\ &\simeq \exp\Big\{\Big[-\mathrm{i}k_1\Big(\frac{x_1-x_0}{\Delta t} - a(x_0)\Big) - \frac{k_1^2}{2}\Big]\Delta t\Big\},\end{aligned}$$

we find again the path-integral expression (let us take this time the continuous limit)

$$\begin{aligned}\rho(x,t) &= \mathcal{N}\int [\mathrm{d}x][\mathrm{d}k]\ \delta(x-x(t)) \\ &\times\ \exp\Big\{-\int_0^t \mathrm{d}\tau\ \Big[\mathrm{i}k(\tau)\dot{x}(\tau) - H(\mathrm{i}k(\tau), x(\tau)\Big]\Big\}\rho_0(x(0)),\end{aligned} \tag{80}$$

where the velocity $\dot{x}(\tau)$ is understood in the Itô sense, as the limit of the second line of Eq. (74). Notice that prescription problems have come all the way down to Eq. (80). Notice in particular the connection between stochastic prescription problem and ordering in Eq. (78). Notice finally that defining the momentum $p = \mathrm{i}k$ allows us to recover the familiar expression

$$\rho(x,t) = \mathcal{N}\int [\mathrm{d}x][\mathrm{d}p]\ \delta(x-x(t))\exp(-S[p,x]) \tag{81}$$

$$S[p,x] = \int_0^t \mathrm{d}\tau\{p(\tau)\dot{x}(\tau) - H(p(\tau), x(\tau))\}. \tag{82}$$

In our special case of a Hamiltonian quadratic in p, carrying out the path integral in $\mathrm{d}p$ leads to the expression

$$\begin{aligned}\rho(x,t) &= \mathcal{N}\int [\mathrm{d}x]\ \delta(x-x(t))\exp(-S[x]) \\ S[x] &= \int_0^t \mathrm{d}\tau\ L(x,\dot{x}) = \frac{1}{2}\int_0^t \mathrm{d}\tau\ (\dot{x}-a(x))^2,\end{aligned} \tag{83}$$

where $S[x]$ and $L(x, \dot{x})$ are the action and the Lagrangian associated with the Hamiltonian H, and we have exploited the relation $\dot{q} = \partial_p H = a(x) + p$.

In all the previous formulae the Itô prescription is adopted. This means that the time integral appearing in the action has all the peculiarities of a stochastic Itô integral. Suppose for instance that the drift in Eq. (49) could be derived from a potential, $a(x) = -V'(x)$, which would mean that the stochastic dynamics is purely dissipative. One of the terms entering $S[x]$ in Eq. (83) would be in this case

$$\int_{t_i}^{t_f} \mathrm{d}\tau\ \dot{x}(\tau) a(x(\tau)) = -\int_{t_i}^{t_f} \mathrm{d}\tau\ \dot{x}(\tau) V'(x(\tau)). \tag{84}$$

The RHS of this equation looks like the total work by the dissipative forces. However, the fundamental theorem of integration does not apply in the case of Itô integrals, for which we have instead

$$\int_{t_i}^{t_f} \mathrm{d}\tau\ \dot{x}(\tau) V'(x(\tau)) = V(x_f) - V(x_i) - \int_{t_i}^{t_f} V''(x(\tau)) \mathrm{d}\tau. \tag{85}$$

This would have the unphysical consequence at equilibrium

$$\int_{t_i}^{t_f} \mathrm{d}\tau\ \langle \dot{x}(\tau) a(x(\tau)) \rangle = \int_{t_i}^{t_f} \langle V''(x(\tau)) \rangle \mathrm{d}\tau \neq 0, \tag{86}$$

meaning that $\langle \dot{x}(\tau) a(x(\tau)) \rangle$ cannot represent the mean power dissipated by the system (which must be zero at equilibrium).

5.1 Perturbation theory and Feynman diagrams

We can treat the integrand $\exp(-S[\mathrm{i}k, x])$ in Eq. (81) as if it where a probability functional density, and introduce the generating functional

$$Z[\eta, \zeta] = \Big\langle \exp\Big\{ -\mathrm{i} \int \mathrm{d}t\ \Big[\eta(t) k(t) + \zeta(t) x(t)\Big]\Big\}\Big\rangle. \tag{87}$$

We can evaluate correlation functions perturbatively in V by sorting out the linear part in the drift of the SDE and then by Taylor expanding Z. Let us indicate $\dot{x} - a(x) = (\partial + \Gamma)x + V(x) = G^{-1}x + V(x)$. The lowest order term in the expansion for Z corresponds to Gaussian statistics

$$\begin{aligned} Z^{(0)}[\eta, \zeta] &= \int [\mathrm{d}k][\mathrm{d}x] \exp\Big\{ -\int \frac{\mathrm{d}\omega}{2\pi} \Big[\mathrm{i}(k_{-\omega} G_\omega^{-1} x_\omega + \mathrm{i}(\eta_\omega k_{-\omega} + \zeta_{-\omega} x_\omega) + \frac{|k_\omega|^2}{2}\Big]\Big\} \\ &= \exp\Big\{ -\int \frac{\mathrm{d}\omega}{2\pi} \Big[\frac{1}{2} C_\omega |\zeta_\omega|^2 - \mathrm{i} G_\omega \zeta_{-\omega} \eta_\omega\Big]\Big\}, \end{aligned} \tag{88}$$

where $C_\omega = |G_\omega|^2 = (\omega^2 + \Gamma^2)^{-1}$ is the Fourier spectrum of x. We find for the bare propagator

$$G_\omega^{\alpha\beta} = -\frac{\partial^2 Z[\eta, \zeta]}{\partial \zeta_{-\omega}^\alpha \partial \eta_\omega^\beta}\Big|_{\zeta=\eta=0} = \mathrm{i} \langle k_{-\omega}^\beta x_\omega^\alpha \rangle, \tag{89}$$